

Classical and quantum analysis of an Einstein-Scalar solution in $2 + 1$ dimensions

O. Gurtug*

T.C. Maltepe University, Faculty of Engineering and Natural Sciences, Istanbul, Turkey[†]

S. Habib Mazharimousavi[‡] and M. Halilsoy[§]

*Department of Physics, Eastern Mediterranean University,
G. Magusa, north Cyprus, Mersin 10, Turkey.*

The classical and quantum properties of a new solution obtained in $2 + 1$ -dimensional gravity coupled with a real scalar field is analyzed in detail. The considered new solution is a one-parameter generalization of a previously known solution. We investigate the solution classically by calculating exact null and null circular geodesics which correspond to a typical particle probe of the corresponding black hole spacetime. The solution admitting naked singularity is investigated within the context of quantum mechanics. The timelike naked singularity is probed with spin-zero and spin-half quantum particles. We show that the curvature singularity in the new solution is stronger and hence in general the spacetime remains quantum singular with respect to a quantum particle probe.

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I. INTRODUCTION

One of the most important motivation to study general relativity in lower dimensions, in particular $2 + 1$ -dimension, is to understand and explore physics in a rather simpler context. Among the others, the well known $2 + 1$ -dimensional study in the literature is the solution obtained by Bañados, Teitelboim and Zanelli (BTZ) [1], which describes black hole solution with a negative cosmological constant. The general properties of the $2 + 1$ -dimensional gravity, particularly the BTZ solution was considered in detail by Carlip in [2]. In this seminal paper, Carlip addressed also the possible relations to quantum gravity via the "microscopic" physics of quantum gravitational states. No doubt that, this work has unveiled the underlying rich structure in $2 + 1$ -dimensional gravity and has motivated many researchers to work along this line.

In the last two decades, $2 + 1$ -dimensional gravity sourced by different matter fields has attracted considerable interest. Einstein-Maxwell extension has been considered both in linear [3] and nonlinear [4–6] electrodynamics. Einstein-scalar and Einstein-Maxwell-scalar extensions are considered in [7, 8] and [9], respectively. Scalar field extension in the form of multiplets is also considered in [10]. In these studies, the solutions admitting black holes are investigated in details in terms of thermodynamic properties. However, solutions admitting naked singularities are studied rarely.

The aim of this paper is to study the classical and the quantum properties of the recently obtained solution that incorporates with a self-interacting real scalar field [10]. This solution is a one parameter generalization of the solution obtained by Schmidt and Singleton (SS) in [8]. The SS solution does not have a horizon and the curvature singularity at $r = 0$ is a typical naked singularity. Another notable feature of the SS solution is that the spatial part of the metric is flat, whereas the temporal part is equivalent to the asymptotic form of AdS spacetime. In contrast to the SS solution, the recently found metric in [10], can possesses black hole or naked singularity depending on the parameters. Furthermore, the spatial part of the metric given in [10] is not flat. Hence, the inclusion of a parameter in the new solution has a drastic effect on the topology of the resulting spacetime.

In this paper; first as a classical analysis, the physical properties of the spacetime geometry of a self-gravitating real scalar field admitting black hole solution will be investigated by considering the geodesics equations. The null and timelike geodesics are obtained exactly by integrating the geodesic equations.

Secondly; as a quantum analysis, the occurrence of naked singularity that the solution given in [10] admits will

*Electronic address: ozaygurtug@maltepe.edu.tr

[†]Department of Physics, Eastern Mediterranean University, G. Magusa, north Cyprus, Mersin 10, Turkey.

[‡]Electronic address: habib.mazhari@emu.edu.tr

[§]Electronic address: mustafa.halilsoy@emu.edu.tr

be investigated. Solutions admitting naked singularities are always undervalued. Hence, our knowledge about their structure / nature is limited. This limited information is in fact natural, because the scale where this singularity forms is smaller than the Planck scale. In this microscopic regime, one would not expect from the classical methods to provide reliable insight. The general belief in these regimes where the curvature of the spacetime becomes enormous is that to adopt the laws of quantum gravity. Consequently, the appropriate method for analyzing the singularities must be found within the framework of quantum theory of gravity. Unfortunately, there is no consistent quantum theory of gravity to be used in singularity analysis yet. In the light of this fact, any method that incorporates with quantum fields / particles for the resolution or persistence of curvature singularities would be considered as a step in the right direction. In the literature, there are alternative methods for this purpose. Loop quantum gravity [11] and string theory [12, 13] are the two important fields in the resolution of singularities. In this paper, an alternative method will be used, so that the motion of quantum fields / particles will be considered in a classical background. In the analysis of the naked singularity, *spin* 0 (scalar) and *spin* 1/2 (fermionic) wave packets obeying the Klein-Gordon and Dirac equations will be used in probing the singularity. Instead of a point particle probe which leads to the notion of *geodesics incompleteness*, the quantum particle probe will be used which leads to the notion of *quantum singularity*.

Organization of the paper is as follows. In Section II we briefly review the new Einstein-scalar solution in 2 + 1-dimensions. Geodesics of a classical particle in the metric admitting black hole is considered in Section III. Quantum probes of spin-0 and spin- $\frac{1}{2}$ are considered for the timelike naked singular version of the solution in Section IV. The paper is completed with Conclusion and Discussion in Section V.

II. THE SOLUTION IN BRIEF

Recently, a new exact radial solution with a self-interacting real scalar field in 2 + 1-dimensional gravity has been introduced in [10]. This new solution in fact constitutes a one-parameter generalization of the solution obtained by Schmidt and Singleton (SS) in [8]. The striking property of the solution presented in [8] is that the spatial part of the metric was flat. However, in the new solution [10], the spatial part is not flat, i.e. the metric tensor component $g_{rr} \neq 1$. In comparison with the solution given by SS, the new solution has a richer structure. One of the distinctive richness is the existence of black holes.

The action and the line element ansatz in Einstein-scalar theory in 2 + 1-dimensions are given by

$$S = \int \sqrt{-g} \left(R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right) d^3x, \quad (1)$$

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + H(r)^2 d\theta^2, \quad (2)$$

respectively in which

$$\phi(r) = \frac{2\alpha}{2 + \alpha^2} \ln r, \quad (3)$$

and

$$V(\phi) = V_0 e^{-\alpha\phi} = V_0 \left(\frac{1}{r} \right)^{\frac{2\alpha^2}{2+\alpha^2}}. \quad (4)$$

Furthermore in [10] it was shown that the solution of the field equations gives

$$A(r) = \frac{V_0 (2 + \alpha^2)^2}{2(\alpha^2 - 4)} r^{\frac{4}{2+\alpha^2}} + C r^{\frac{\alpha^2}{2+\alpha^2}} \quad (5)$$

and

$$H(r) = r^{\frac{4}{2+\alpha^2}}. \quad (6)$$

In these solutions $\alpha \neq \pm 2$ and V_0 are two real parameters and C is an integration constant.

Finally, the line element in terms of the parameters can be written as

$$ds^2 = - \left(\frac{V_0 (\alpha^2 + 2)^2}{2(\alpha^2 - 4)} r^{\frac{4}{2+\alpha^2}} + C r^{\frac{\alpha^2}{2+\alpha^2}} \right) dt^2 + \frac{dr^2}{\left(\frac{V_0 (\alpha^2 + 2)^2}{2(\alpha^2 - 4)} r^{\frac{4}{2+\alpha^2}} + C r^{\frac{\alpha^2}{2+\alpha^2}} \right)} + r^{\frac{4}{2+\alpha^2}} d\theta^2 \quad (7)$$

which is in general highly dependent on the three parameters α , C and V_0 . Depending on the choice of these parameters, the general solution admits black holes or naked singularities. For instance, the black hole solution is possible if one sets $\alpha = \sqrt{2}$ and $C < 0$ with $V_0 < 0$ where a quasilocal mass evaluated from the Brown-York [14] formalism is given by

$$M_{QL} = \frac{|C|}{8}. \quad (8)$$

Note that if we take $\alpha = \sqrt{2}$ we obtain the following line element

$$ds^2 = -(C\sqrt{r} - 4V_0r) dt^2 + \frac{dr^2}{(C\sqrt{r} - 4V_0r)} + r d\theta^2. \quad (9)$$

After redefining the radial coordinate as $r = \rho^2$, the metric (9) becomes

$$ds^2 = -\rho(C - 4V_0\rho) dt^2 + \frac{4\rho}{(C - 4V_0\rho)} d\rho^2 + \rho^2 d\theta^2. \quad (10)$$

Before going further let us justify our choice of α by the fact that $\frac{\partial\phi}{\partial\alpha} = 0$, and $\frac{\partial^2\phi}{\partial\alpha^2} < 0$, at $\alpha^2 = 2$, which provides the maximum effect for this specific α . The resulting metric given in equation (10), can be interpreted as a one-parameter generalization (namely, with C) of the solution obtained by SS. One can easily recover the solution in SS, if $C = 0$, $V_0 = -1$ [8]. Obviously, the resulting metric (10) admits naked singularities if $C > 0$, $V_0 < 0$ or $C < 0$, $V_0 > 0$. One of the scope of this paper is to investigate the nature of this naked singularity in view of quantum mechanics.

III. GEODESIC MOTION OF A CLASSICAL PARTICLE

As we mentioned in previous section, the general solution for $\alpha^2 < 4$ admits black hole with a general line element given by [15]

$$ds^2 = -f(r) dt^2 + \frac{r^{\alpha^2}}{f(r)} dr^2 + r^2 d\theta^2 \quad (11)$$

in which

$$f(r) = \frac{4}{4 - \alpha^2} \frac{r^2}{\ell^2} \left(1 - \left(\frac{r_h}{r} \right)^{\frac{4 - \alpha^2}{2}} \right) \quad (12)$$

with

$$r_h = \left(\frac{\ell^2 (4 - \alpha^2) M_{BTZ}}{4} \right)^{\frac{2}{4 - \alpha^2}}. \quad (13)$$

Herein M_{BTZ} guarantees that in the limit $\alpha \rightarrow 0$ we get the corresponding BTZ solution with the proper mass [15]. In addition to these, in terms of the parameter used in previous section one finds

$$\ell^2 = \frac{8}{|V_0|(\alpha^2 + 2)^2} \quad (14)$$

and

$$r_h = \left(\frac{2|C|(4 - \alpha^2)}{|V_0|(\alpha^2 + 2)^2} \right)^{\frac{2}{4 - \alpha^2}} \quad (15)$$

provided both C and V_0 are negative.

The Lagrangian of a classical test particle with unit mass is given by

$$L = \frac{1}{2} \left(-f(r) \dot{t}^2 + \frac{r^{\alpha^2}}{f(r)} \dot{r}^2 + r^2 \dot{\theta}^2 \right) \quad (16)$$

in which a dot stands for the derivative with respect to the proper time τ . The normalization condition, in other hands, imposes $U^\mu U_\mu = \epsilon$ or

$$-f(r)\dot{t}^2 + \frac{r^{\alpha^2}}{f(r)}\dot{r}^2 + r^2\dot{\theta}^2 = \epsilon \quad (17)$$

in which U^μ is the four-velocity and $\epsilon = 0, \pm 1$ for null, timelike / spacelike geodesics. The Euler Lagrange equation can be found by introducing

$$P_0 = -E = \frac{\partial L}{\partial \dot{t}} = -f\dot{t} \quad (18)$$

and

$$P_2 = J = \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta} \quad (19)$$

in which E is the energy of the particle and J the angular momentum. Inserting (19) and (18) into (17), we find the following

$$r^2 = \left(\epsilon + \frac{E^2}{f} - \frac{J^2}{r^2} \right) \frac{f}{r^{\alpha^2}}. \quad (20)$$

A combination of (19) and (20) admits

$$\left(\frac{dr}{d\theta} \right)^2 = \left(\frac{\epsilon}{J^2} + \frac{E^2}{J^2 f} - \frac{1}{r^2} \right) \frac{f}{r^{\alpha^2-4}}. \quad (21)$$

A. Null geodesics

Setting $\epsilon = 0$ in (21) one finds

$$\left(\frac{dr}{d\theta} \right)^2 = \frac{\xi^2 r^2 - f}{r^{\alpha^2-2}} \quad (22)$$

in which $\xi^2 = \frac{E^2}{J^2}$. Considering (12) the latter equation reads

$$\left(\frac{dr}{d\theta} \right)^2 = \left(\xi^2 - \frac{4}{(4-\alpha^2)\ell^2} \right) r^{4-\alpha^2} + \frac{4}{(4-\alpha^2)\ell^2} (rr_h)^{\frac{4-\alpha^2}{2}}. \quad (23)$$

To proceed further we set $\alpha^2 = 2$ resulting in

$$\left(\frac{dr}{d\theta} \right)^2 = \left(\xi^2 - \frac{2}{\ell^2} \right) r^2 + \frac{2}{\ell^2} r r_h \quad (24)$$

which upon introducing the new variables $\tilde{r} = \frac{\ell^2}{2r_h} \left(\xi^2 - \frac{2}{\ell^2} \right) r$ and $\tilde{\theta} = \theta \sqrt{\xi^2 - \frac{2}{\ell^2}}$ this yields

$$\left(\frac{d\tilde{r}}{d\tilde{\theta}} \right)^2 = \tilde{r}^2 + \tilde{r}. \quad (25)$$

This admits a solution given by

$$\tilde{r}_\pm = \frac{e^{\pm(\tilde{\theta} + \tilde{C})}}{2} - \frac{1}{2} + \frac{e^{\mp(\tilde{\theta} + \tilde{C})}}{8} \quad (26)$$

in which \tilde{C} is an integration constant.

Finally we add that the inverse transformation of the variables yield a complete solution expressed by

$$r_{\pm} = \frac{e^{\pm\sqrt{\xi^2 - \frac{2}{\ell^2}}(\theta+\eta)} - 1 + \frac{1}{4}e^{\mp\sqrt{\xi^2 - \frac{2}{\ell^2}}(\theta+\eta)}}{\ell^2 \left(\xi^2 - \frac{2}{\ell^2}\right)} r_h \quad (27)$$

where $\tilde{C} = \eta\sqrt{\xi^2 - \frac{2}{\ell^2}}$ provided $\xi^2 - \frac{2}{\ell^2} > 0$.

In null geodesics when $\epsilon = 0$ the coordinate time t equation of the position r is given by the combination of Eqs. (20) and (18). Hence, one obtains

$$\frac{dr}{dt} = \pm \sqrt{1 - \frac{J^2 f}{E^2 r^2}} \frac{f}{r^{\alpha^2/2}} \quad (28)$$

where for $\alpha^2 = 2$ and f given by (12) we find

$$\frac{dr}{dt} = \pm \sqrt{1 - \frac{2J^2}{E^2 \ell^2} + \frac{2J^2}{E^2 \ell^2} \frac{r_h}{r}} \frac{2}{\ell^2} (r - r_h) \quad (29)$$

which upon a change of variable of the form $r = \frac{r_h}{u}$ becomes

$$\frac{du}{dt} = \mp \sqrt{1 - \frac{2J^2}{E^2 \ell^2} + \frac{2J^2}{E^2 \ell^2} u} \frac{2u}{\ell^2} (1 - u). \quad (30)$$

This equation admits an exact solution given by

$$\mp \frac{t - t_0}{\ell^2} = \tanh^{-1} \frac{\sqrt{1 - \frac{2J^2}{E^2 \ell^2} + \frac{2J^2}{E^2 \ell^2} u} - \sqrt{1 - \frac{2J^2}{E^2 \ell^2} + \frac{2J^2}{E^2 \ell^2} u_0}}{1 - \sqrt{1 - \frac{2J^2}{E^2 \ell^2} + \frac{2J^2}{E^2 \ell^2} u} \sqrt{1 - \frac{2J^2}{E^2 \ell^2} + \frac{2J^2}{E^2 \ell^2} u_0}} + \quad (31)$$

$$\frac{1}{\sqrt{1 - \frac{2J^2}{E^2 \ell^2}}} \tanh^{-1} \left(\frac{\sqrt{1 - \frac{2J^2}{E^2 \ell^2}} \left(\sqrt{1 - \frac{2J^2}{E^2 \ell^2} + \frac{2J^2}{E^2 \ell^2} u} - \sqrt{1 - \frac{2J^2}{E^2 \ell^2} + \frac{2J^2}{E^2 \ell^2} u_0} \right)}{1 - \frac{2J^2}{E^2 \ell^2} - \sqrt{1 - \frac{2J^2}{E^2 \ell^2} + \frac{2J^2}{E^2 \ell^2} u} \sqrt{1 - \frac{2J^2}{E^2 \ell^2} + \frac{2J^2}{E^2 \ell^2} u_0}} \right) \quad (32)$$

in which $u_0 = u(t = t_0)$. As $u = \frac{r_h}{r}$ is smaller than 1 i.e., $u < 1$, so is u_0 which imply that $1 > \frac{2J^2}{E^2 \ell^2}$. At exactly $\frac{2J^2}{E^2 \ell^2} = 1$ the solution is obtained to be

$$\mp \frac{t - t_0}{\ell^2} = -(x - x_0) + \ln \left(\frac{(x + 1)(x_0 - 1)}{(x - 1)(x_0 + 1)} \right) \quad (33)$$

in which $x = \left(\frac{r}{r_h}\right)^2 = \frac{1}{u^2} > 1$ and $x_0 = x(t = t_0)$. Note that the negative branch corresponds to $\frac{dr}{dt} > 0$ and vice versa. From the latter equation we comment that in the negative branch solution the null particle escapes from the black hole and in an infinite time interval it reaches to infinity. In contrary, the positive branch solution approaches to the horizon in an infinite time interval. These are shown in Fig. 1.

B. Null circular geodesics

In circular motion we have $r = r_c$ and is constant. This together with (24) we find

$$\left(\xi^2 - \frac{2}{\ell^2}\right) r_c + \frac{2}{\ell^2} r_h = 0. \quad (34)$$

This equation admits a solution for r_c given by

$$r_c = \frac{1}{1 - \frac{\ell^2 E^2}{2J^2}} r_h \quad (35)$$

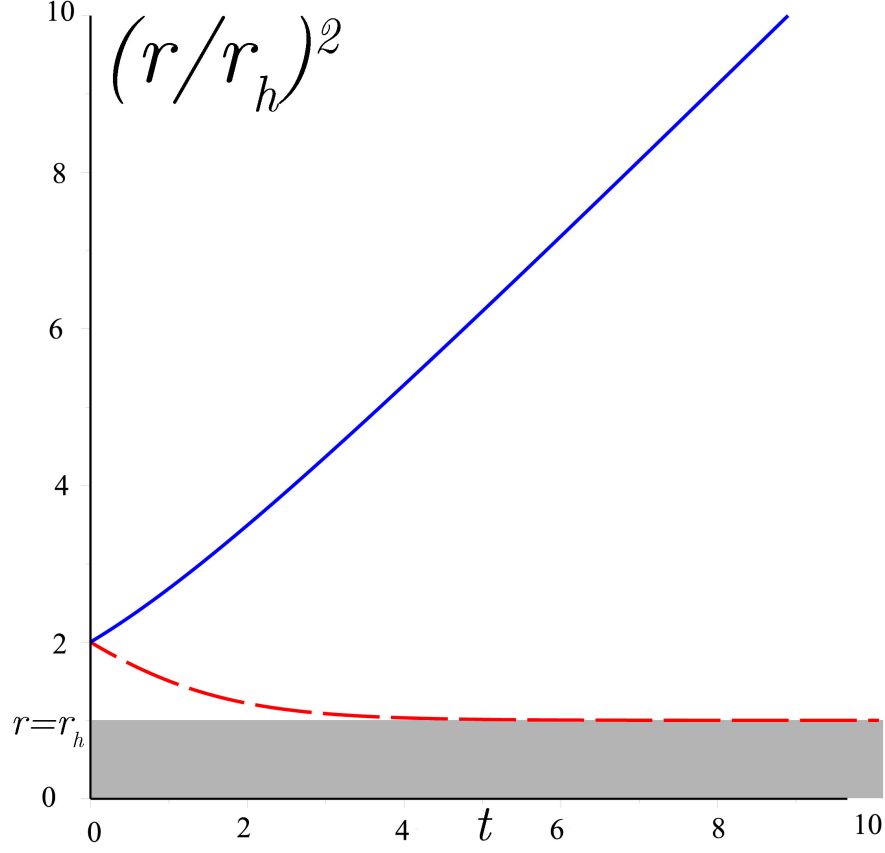


FIG. 1: $x(t) = \left(\frac{r}{r_h}\right)^2$ versus t/ℓ^2 for positive (Red-Dash) and negative (Blue-Solid) branch solutions corresponding to the negative and positive initial velocity respectively and $\frac{2J^2}{E^2\ell^2} = 1$.

which for $1 > \frac{\ell^2 E^2}{2J^2}$ the solution is positive and larger than the radius of event horizon. Therefore for null particle with given $\frac{E^2}{J^2}$ there exist only one single possible circular orbit whose radius is obtained as (35). Looking at the master radial equation for the null particle with affine parameter λ

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{E^2}{r^2} - \frac{J^2 f}{r^4} \quad (36)$$

and rewriting it as

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{2J^2}{\ell^2} \frac{1}{r^2} \left(\frac{r_h}{r} - \frac{r_h}{r_c}\right) \quad (37)$$

one can find

$$\frac{d^2 r}{d\lambda^2} = \frac{J^2}{\ell^2} \frac{2r_h}{r^3 r_c} \left(1 - \frac{3r_c}{2r}\right) \quad (38)$$

which is clearly negative at $r = r_c$, indicating that the circular motion is unstable.

C. Null radial geodesics

For the radial motion of a null particle we set $J = 0$ in (29) which yields

$$\frac{dr}{dt} = \pm \frac{2}{\ell^2} (r - r_h) \quad (39)$$

whose exact solution is given by

$$\frac{r}{r_h} = 1 + \left(\frac{r_0}{r_h} - 1 \right) e^{\pm \frac{2}{\ell^2} (t-t_0)} \quad (40)$$

in which $r(t = t_0) = r_0$. In the case of outward initial velocity the radial position of the null particle increases as fast as the exponential of time and for the inward initial velocity it approaches asymptotically to the horizon.

IV. QUANTUM PROBE OF THE SOLUTION ADMITTING NAKED SINGULARITY

Spacetime singularities constitutes one of the important predictions of the Einstein's theory of relativity. It describes the location in the fabric of spacetime in which the physical quantities like tidal forces, gravitational fields and energy densities becomes unbounded. Or, in other words, it is defined as the "end point" for the time evolution of the timelike and null geodesics. According to the Penrose's weak cosmic censorship hypothesis, singularities must be covered by horizon(s), for the deterministic nature of the theory of general relativity. However, there are some situations such that the solutions to the Einstein's equations admit singularities that are not hidden by horizon(s) and becomes visible to the asymptotic observers. Singularities in this kind are called naked singularities and violates the cosmic censorship hypothesis of Penrose. A question has been asked: how could this singularity problem in general relativity be resolved ?.

So far, it seems that the most promising method for the resolution of the singularities is the quantum theory of gravity. The reason for this is the scales where these singularities are developed, and hence, classical general relativity is expected to be replaced by quantum theory of gravity. However, a complete quantum theory of gravity is not available yet. For this reason, an alternative method which was developed by Horowitz and Marolf (HM) [16] following the work of Wald [17] will be used in this paper. In order to probe the naked singularity, quantum particles with $spin = 0$ (scalar) and $spin = 1/2$ (fermion) comply with Klein - Gordon and Dirac equations will be used, respectively.

The main theme of the HM criterion which can be applied only to static spacetimes having timelike naked singularities is summarized as follows: The crucial idea is to split the spatial and time part of the Klein-Gordon equation and write it in the form of

$$\frac{\partial^2 \psi}{\partial t^2} = -A\psi, \quad (41)$$

where A is the spatial wave operator. Note that, this operator is a symmetric and positive on the Hilbert space \mathcal{H} . According to the HM, the singular character of the spacetime with respect to quantum probe is characterized by investigating whether the spatial part of the operator A has a unique self - adjoint extension (i.e. essentially self - adjoint) in the entire space or not. If the extension is unique, it is said that the space is quantum mechanically regular. In order to clarify this point, consider the Klein-Gordon equation for a free particle that satisfies

$$i \frac{d\psi}{dt} = \sqrt{A_E} \psi, \quad (42)$$

whose solution is

$$\psi(t) = e^{-it\sqrt{A_E}} \psi(0), \quad (43)$$

in which A_E denotes the extension of the operator A . If A has not a unique self - adjoint extension, then the future time evolution of the wave function (42) is ambiguous and HM criterion defines the spacetime as quantum mechanically singular. The essential self-adjointness of the operator A , can be verified by considering solutions of the equation

$$A^* \psi \pm i\psi = 0, \quad (44)$$

(with A^* denoting the adjoint of A) and showing that the solutions of (44), do not belong to Hilbert space \mathcal{H} (we refer to [18, 21] for detailed mathematical analysis). This will be achieved by defining the function space on each $t = \text{constant}$ hypersurface Σ as $\mathcal{H} = \{R : \|R\| < \infty\}$ with the following norm given for the metric (10) as,

$$\|R\|^2 = \int_0^\infty \sqrt{\frac{g_{\rho\rho} g_{\theta\theta}}{g_{tt}}} |R|^2 d\rho = \int_0^\infty \frac{\rho}{A(\rho)} |R|^2 d\rho. \quad (45)$$

A. Klein-Gordon Fields

The timelike naked singularity for the metric (10) will be probed with massive scalar particles (bosons) satisfying the Klein-Gordon equation

$$\left(\frac{1}{\sqrt{g}} \partial_\mu [\sqrt{g} g^{\mu\nu} \partial_\nu] - \tilde{m}^2 \right) \psi = 0, \quad (46)$$

in which \tilde{m} is the mass of the scalar particle. Equation (46) for the metric (10) is given below by splitting temporal and spatial part as

$$\frac{\partial^2 \psi}{\partial t^2} = - \left\{ - \left(\frac{C - 4V_0\rho}{2} \right)^2 \frac{\partial^2 \psi}{\partial \rho^2} - \left[\frac{(C - 4V_0\rho)(C - 8V_0\rho)}{4\rho} \right] \frac{\partial \psi}{\partial \rho} - \left(\frac{C - 4V_0\rho}{\rho} \right) \frac{\partial^2 \psi}{\partial \theta^2} - \tilde{m}^2 \psi \right\}. \quad (47)$$

By writing the Klein-Gordon equation in the form of (41), one can easily write the spatial part of the wave operator as

$$A = - \left(\frac{C - 4V_0\rho}{2} \right)^2 \frac{\partial^2}{\partial \rho^2} - \left[\frac{(C - 4V_0\rho)(C - 8V_0\rho)}{4\rho} \right] \frac{\partial}{\partial \rho} - \left(\frac{C - 4V_0\rho}{\rho} \right) \frac{\partial^2}{\partial \theta^2} - \tilde{m}^2 \quad (48)$$

The problem now is to count the number of extensions of the operator A . This is done by using the concept of deficiency indices discovered by Weyl [22] and generalized by von Neumann [23]. If there are no square integrable ($L^2(0, \infty)$) solutions (i.e. with deficiency indices $n_+ = n_- = 0$) in the entire space, the operator A possesses a unique self-adjoint extension and it is called essentially self-adjoint. Consequently, the method to find a sufficient condition for the operator A to be essentially self-adjoint is to investigate the solutions satisfying equation (44) that do not belong to the Hilbert space \mathcal{H} . Using separation of variables, $\psi = R(\rho)Y(\theta)$, equation (44) yields the following radial equation

$$\frac{\partial^2 R(\rho)}{\partial \rho^2} + \left[\frac{C - 8V_0\rho}{\rho(C - 4V_0\rho)} \right] \frac{\partial R(\rho)}{\partial \rho} + \left(\frac{4}{C - 4V_0\rho} \right) \left[\frac{\tilde{m}^2 \pm i}{C - 4V_0\rho} - \frac{c^2}{\rho} \right] R(\rho) = 0, \quad (49)$$

in which c denotes the separation constant. Because of the complexity in finding exact analytic solution to Eq. (49), we study the behavior of $R(\rho)$ near $\rho \rightarrow 0$ and $\rho \rightarrow \infty$.

1. The case when $\rho \rightarrow 0$

The behaviour of equation (49) when $\rho \rightarrow 0$ is given by

$$\frac{\partial^2 R(\rho)}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial R(\rho)}{\partial \rho} - \frac{4c^2}{\rho C_2} R(\rho) = 0 \quad (50)$$

whose solution is given in terms of first kind modified Bessel functions as

$$R(\rho) = a_1 I_0(4c\sqrt{\frac{\rho}{C}}) + a_2 K_0(4c\sqrt{\frac{\rho}{C}}), \quad (51)$$

in which a_1 and a_2 are the integration constants. The square integrability of the solution (51) is checked by calculating the squared norm defined in Eq. (45). In the limiting case of $\rho \rightarrow 0$, the squared norm reduces to the following form

$$\|R(\rho)\|^2 \sim \int_0^{const.} \rho |R(\rho)|^2 d\rho. \quad (52)$$

Calculations have revealed that the squared norm $\|R(\rho)\|^2 < \infty$, indicating that the solution is square integrable and belongs to the Hilbert space.

2. The case when $\rho \rightarrow \infty$

In the case when $\rho \rightarrow \infty$, the Eq. (49) simplifies to,

$$\frac{\partial^2 R(\rho)}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial R(\rho)}{\partial \rho} + \frac{\alpha}{\rho^2} R(\rho) = 0, \quad (53)$$

whose solution is

$$R(\rho) = \frac{a_3 \rho^{\frac{\gamma}{2}} + a_4 \rho^{-\frac{\gamma}{2}}}{\sqrt{\rho}}, \quad (54)$$

in which $\gamma = \sqrt{1 - 4\alpha}$, and a_3 and a_4 are the integration constants.

The square integrability of the solution (54) is checked by calculating the squared norm defined in equation (45) in the limiting case of the metric (10) when $\rho \rightarrow \infty$, which is given by

$$\|R(\rho)\|^2 \sim \int_{const.}^{\infty} |R(\rho)|^2 d\rho. \quad (55)$$

The analysis has indicated that depending on the values of the complex constant α together with the integration constants a_3 and a_4 , there are some specific solutions that fails to satisfy square integrability condition (i.e., $\|R(\rho)\|^2 \rightarrow \infty$). However, for a generic case the solution is square integrable and belongs to the Hilbert space.

The method of defining whether the operator A has a unique self-adjoint extension (or essentially self-adjoint) or not is to investigate the solution of Eq. (49) in the entire space $(0, \infty)$ and count the number of solutions that do not belong to the Hilbert space. In other words, if there is one solution that fails to be square integrable for the entire space then the operator A is said to be essentially self-adjoint. Our analysis has shown that although the behaviour of Eq. (49), when $r \rightarrow \infty$, admits solution that is not square integrable, generically, it is square integrable. Hence, the operator A is not essentially self-adjoint and the future time evolution of the quantum particles/waves can not be predicted uniquely. Consequently, the classical naked singularity persists and remains quantum mechanically singular when probed with massive bosons described by the Klein-Gordon equation.

B. Dirac Fields

The timelike naked singularity of the considered spacetime is probed also with spinorial fields ($spin = 1/2$) obeying the Dirac equation. The adopted method for the solution of the Dirac equation in $2+1$ -dimensional curved geometry was given in [24]. This method has been successfully used in earlier studies along this line [24, 28]. The Dirac equation in $2+1$ dimensional curved geometry for a free particle with mass m is given by,

$$i\sigma^\mu(x) [\partial_\mu - \Gamma_\mu(x)] \Psi(x) = m\Psi(x), \quad (56)$$

where $\Gamma_\mu(x)$ is the spinorial affine connection given by

$$\Gamma_\mu(x) = \frac{1}{4} g_{\lambda\alpha} \left[e_{\nu,\mu}^{(i)}(x) e_{(i)}^\alpha(x) - \Gamma_{\nu\mu}^\alpha(x) \right] s^{\lambda\nu}(x), \quad (57)$$

with

$$s^{\lambda\nu}(x) = \frac{1}{2} [\sigma^\lambda(x), \sigma^\nu(x)]. \quad (58)$$

Since the fermions have only one spin polarization in $2+1$ -dimension, the Dirac matrices $\gamma^{(j)}$ can be expressed in terms of Pauli spin matrices $\sigma^{(i)}$ so that

$$\gamma^{(j)} = (\sigma^{(3)}, i\sigma^{(1)}, i\sigma^{(2)}), \quad (59)$$

where the Latin indices represent internal (local) frame. In this way,

$$\left\{ \gamma^{(i)}, \gamma^{(j)} \right\} = 2\eta^{(ij)} I_{2 \times 2}, \quad (60)$$

where $\eta^{(ij)}$ is the Minkowski metric in 2 + 1-dimension and $I_{2 \times 2}$ is the identity matrix. The coordinate dependent metric tensor $g_{\mu\nu}(x)$ and matrices $\sigma^\mu(x)$ are related to the triads $e_\mu^{(i)}(x)$ by

$$\begin{aligned} g_{\mu\nu}(x) &= e_\mu^{(i)}(x) e_\nu^{(j)}(x) \eta_{(ij)}, \\ \sigma^\mu(x) &= e_{(i)}^\mu \gamma^{(i)}, \end{aligned} \quad (61)$$

where μ and ν stand for the external (global) indices. The suitable triads for the metric (10) is given by,

$$e_\mu^{(i)}(t, \rho, \theta) = \text{diag} \left(\sqrt{\rho(C - 4V_0\rho)}, 2\sqrt{\frac{\rho}{C - 4V_0\rho}}, \rho \right), \quad (62)$$

The coordinate dependent gamma matrices and the spinorial affine connection are given by

$$\begin{aligned} \sigma^\mu(x) &= \left(\frac{\sigma^{(3)}}{\sqrt{\rho(C - 4V_0\rho)}}, \frac{i\sigma^{(1)}}{2} \sqrt{\frac{C - 4V_0\rho}{\rho}}, \frac{i\sigma^{(2)}}{\rho} \right), \\ \Gamma_\mu(x) &= \left(-\frac{(C - 8V_0\rho)\sigma^{(2)}}{8\rho}, 0, \frac{i\sigma^{(3)}}{4} \sqrt{\frac{C - 4V_0\rho}{\rho}} \right). \end{aligned} \quad (63)$$

Now, for the spinor

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (64)$$

the Dirac equation can be written as

$$\frac{i}{\sqrt{\rho\Sigma(\rho)}} \frac{\partial \psi_1}{\partial t} - \sqrt{\frac{\Sigma(\rho)}{4\rho}} \frac{\partial \psi_2}{\partial \rho} + \frac{i}{\rho} \frac{\partial \psi_2}{\partial \theta} + \left(\frac{C - 8V_0\rho}{8\rho\sqrt{\rho\Sigma(\rho)}} - \sqrt{\frac{\Sigma(\rho)}{16\rho^3}} \right) \psi_2 = m\psi_1 \quad (65)$$

and

$$-\frac{i}{\sqrt{\rho\Sigma(\rho)}} \frac{\partial \psi_2}{\partial t} - \sqrt{\frac{\Sigma(\rho)}{4\rho}} \frac{\partial \psi_1}{\partial \rho} - \frac{i}{\rho} \frac{\partial \psi_1}{\partial \theta} + \left(\frac{C - 8V_0\rho}{8\rho\sqrt{\rho\Sigma(\rho)}} - \sqrt{\frac{\Sigma(\rho)}{16\rho^3}} \right) \psi_1 = m\psi_2 \quad (66)$$

in which $\Sigma(\rho) = C - 4V_0\rho$. The coupled equations (65) and (66), can be solved by employing the following ansatz for the positive frequency solutions which was also used in the studies [24, 28]

$$\Psi_{n,E}(t, x) = \begin{pmatrix} R_{1n}(\rho) \\ R_{2n}(\rho)e^{i\theta} \end{pmatrix} e^{in\theta} e^{-iEt}. \quad (67)$$

In terms of radial function, equations (65) and (66) transforms into,

$$R'_{1n}(\rho) + \mathcal{A}(\rho) R_{1n}(\rho) + \mathcal{B}(\rho) R_{2n}(\rho)e^{i\theta} = 0 \quad (68)$$

and

$$R'_{2n}(\rho) + \mathcal{C}(\rho) R_{2n}(\rho) + \mathcal{D}(\rho) R_{1n}(\rho)e^{-i\theta} = 0. \quad (69)$$

Equations (68) and (69), can be cast into the following decoupled equations,

$$R''_{1n}(\rho) + \left\{ \mathcal{A}(\rho) + \mathcal{C}(\rho) - \frac{\mathcal{B}'(\rho)}{\mathcal{B}(\rho)} \right\} R'_{1n}(\rho) + \left\{ \mathcal{A}(\rho) \mathcal{C}(\rho) - \mathcal{B}(\rho) \mathcal{D}(\rho) + \mathcal{B}(\rho) \left(\frac{\mathcal{A}(\rho)}{\mathcal{B}(\rho)} \right)' \right\} R_{1n}(\rho) = 0 \quad (70)$$

and

$$R''_{2n}(\rho) + \left\{ \mathcal{A}(\rho) + \mathcal{C}(\rho) - \frac{\mathcal{D}'(\rho)}{\mathcal{D}(\rho)} \right\} R'_{2n}(\rho) + \left\{ \mathcal{A}(\rho) \mathcal{C}(\rho) - \mathcal{B}(\rho) \mathcal{D}(\rho) + \mathcal{D}(\rho) \left(\frac{\mathcal{C}(\rho)}{\mathcal{D}(\rho)} \right)' \right\} R_{2n}(\rho) = 0 \quad (71)$$

where

$$\begin{aligned} \mathcal{A}(\rho) &= \frac{C}{4\rho\Sigma(\rho)} - \frac{2n}{\sqrt{\rho\Sigma(\rho)}}, & \mathcal{B}(\rho) &= \frac{2m\sqrt{\rho}}{\sqrt{\Sigma(\rho)}} + \frac{2E}{\Sigma(\rho)}, \\ \mathcal{C}(\rho) &= \frac{C}{4\rho\Sigma(\rho)} - \frac{2(n+1)}{\sqrt{\rho\Sigma(\rho)}}, & \mathcal{D}(\rho) &= \frac{2m\sqrt{\rho}}{\sqrt{\Sigma(\rho)}} - \frac{2E}{\Sigma(\rho)}. \end{aligned} \quad (72)$$

Note that the prime denote the derivative with respect to ρ . The solutions of the radial part of the Dirac equations should be analyzed for a unique self-adjoint extension for all space $L^2(0, \infty)$. Because of the complexity of Eqs. (70,71), the behaviour of the Dirac fields will be investigated near $\rho \rightarrow 0$ and $\rho \rightarrow \infty$.

1. The case when $\rho \rightarrow 0$

The behaviour of the radial part of the Dirac equations (70) and (71) near $\rho \rightarrow 0$ is given by

$$R''_{in}(\rho) + \frac{1}{2\rho} R'_{in}(\rho) - \frac{3}{16\rho^2} R_{in}(\rho) = 0 \quad \text{where} \quad i = 1, 2 \quad (73)$$

whose solutions are

$$R_{in}(\rho) = a_5 r^{(3/4)} + a_6 r^{(1/4)} \quad \text{where} \quad i = 1, 2 \quad (74)$$

The square integrability is checked with the following norm written for the case of $\rho \rightarrow 0$,

$$\|R_{in}(\rho)\|^2 \sim \int_0^{const.} \rho |R_{in}(\rho)|^2 d\rho \quad \text{where} \quad i = 1, 2 \quad (75)$$

The analysis has revealed that the squared norm is finite (i.e. $\|R(\rho)\|^2 < \infty$) indicating that the solution is belonging to the Hilbert space.

2. The case when $\rho \rightarrow \infty$

The behaviour of the radial part of the Dirac equations (73) and (74) near $\rho \rightarrow \infty$ is given by

$$R''_{in}(\rho) - \frac{1}{\rho\sqrt{|V_0|}} R'_{in}(\rho) - \frac{2m^2}{|V_0|} R_{in}(\rho) = 0 \quad \text{where} \quad i = 1, 2 \quad (76)$$

whose solutions are

$$R_{in}(\rho) = a_7 r^\kappa I_\kappa \left(\frac{m\sqrt{2}}{\sqrt{|V_0|}} r \right) + a_8 r^\kappa K_\kappa \left(\frac{m\sqrt{2}}{\sqrt{|V_0|}} r \right) \quad \text{where} \quad i = 1, 2 \quad (77)$$

in which $\kappa = \frac{1+\sqrt{|V_0|}}{2\sqrt{|V_0|}}$, $a_5 \dots a_8$ are the integration constants. The square integrability is checked with the following norm written for the case of $\rho \rightarrow \infty$,

$$\|R_{in}(\rho)\|^2 \sim \int_{const.}^\infty |R_{in}(\rho)|^2 d\rho \quad \text{where} \quad i = 1, 2 \quad (78)$$

The analysis has shown that the solution fails to satisfy square integrability condition if one choose $a_7 \neq 0$ but $a_8 = 0$. On the other hand, the solution is square integrable if we take $a_7 = 0$ but $a_8 \neq 0$. The generic conclusion from this analysis is that the classical naked singularity remains quantum mechanically singular when probed with spinor fields obeying Dirac equations.

V. CONCLUSION AND DISCUSSION

In this paper, we analyzed in details, the recently obtained solution in $2 + 1$ – dimensional gravity that describes a radial solution coupled with a real scalar field [10]. The notable feature of this solution is that depending on the values of the parameters, one may get black hole or naked singular solutions. The black hole properties are analyzed classically by calculating null and null circular geodesics. On the other hand, quantum analysis is carried out by probing the timelike naked singularity that forms for particular values of the parameters with quantum particles. We investigate the geodesic motion of a massless particle in the gravitational field of a black hole. The results reveal the typical behaviour of null geodesics around a black hole. The significance of the naked singularity is analyzed by sending quantum particles to the timelike naked singularity that obeys the Klein-Gordon and Dirac equations. Our analysis have indicated that, in the generic case, the singularity is persistent and remains quantum mechanically singular with respect to the quantum particle probes. In our earlier study [28], we investigated the quantum singularity structure of the SS solution. In that study, we showed that the classical singularity remains singular against bosonic/scalar probe, but it becomes quantum regular with respect to the fermionic probe. As stated in Section II, the solution analyzed in this article is a one-parameter generalization of the SS solution and this parameter has an influence on the curvature of the resulting spacetime. The square of Ricci tensor in the SS solution was given in [8], as $R_{\mu\nu}R^{\mu\nu} = \frac{2}{r^4}$, whereas in the considered solution it is calculated and given by $R_{\mu\nu}R^{\mu\nu} = \frac{2V_0^2}{r^4} + \frac{C^2}{r^6}$. The presence of the parameter C increases the rate of divergence of Ricci tensor. This means that the singularity in the considered solution [10] is stronger than the singularity of the SS solution. As a consequence, fermionic/scalar probe of the singularity does not help to heal the singularity and remains quantum singular.

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- [1] M. Bañados, C. Teitelboim and J. Zanelli, *Phys. Rev. Lett.* **69**, 1849 (1992).
 - [2] S. Carlip, *Class. Quant. Grav.*, **12**, 2853, (1995).
 - [3] C. Martinez, C. Teitelboim and J. Zanelli, *Phys. Rev. D* **61**, 104013 (2000).
 - [4] M. Cataldo, N. Cruz, S. D. Campo, and A. Garcia, *Phys. Lett. B* **484**, 154, (2000).
 - [5] O. Gurtug, S. H. Mazharimousavi and M. Halilsoy, *Phys. Rev. D* **85**, 104004 (2012).
 - [6] M. Cataldo and A. Garcia, *Phys. Rev. D* **61**, 084003 (2000).
 - [7] T. Birkandan and M. Hortaçsu, *Gen. Rel. Grav.*, **35**, 457-466, (2003).
 - [8] H. J. Schmidt and D. Singleton, *Phys. Lett. B* **721**, 294-298 (2013).
 - [9] W. Xu and L. Zhao, *Phys. Rev. D* **87**, 124008 (2013).
 - [10] S. Habib. Mazharimousavi and M. Halilsoy, *Eur. Phys. J. C.*, **76**, 458 (2016).
 - [11] A. Ashtekar, *J. Phys. Conf. Ser.* **189**, 012003 (2009).
 - [12] G. T. Horowitz, *New J. Phys.* **7**, 201 (2005).
 - [13] M. Natsume, arXiv:gr-qc/0108059.
 - [14] J. D. Brown and J. W. York, *Phys. Rev. D* **47**, 1407 (1993).
 - [15] S. H. Mazharimousavi, Z. Amirabi and M. Halilsoy, arXiv:1702.06021” *Thin-shell wormholes in 2+1-dimensional Einstein-Scalar Theory*” *Mod. Phys. Lett. A* (2017) in press.
 - [16] G. T. Horowitz and D. Marolf, *Phys. Rev. D* **52**, 5670 (1995).
 - [17] R. M. Wald, *J. Math. Phys. (N.Y.)* **21**, 2082 (1980).
 - [18] M. Reed and B. Simon, *Functional Analysis*, (Academic Press, New York, 1980).
 - [19] M. Reed and B. Simon, *Fourier Analysis and Self-Adjointness*, (Academic Press, New York, 1975).
 - [20] R. D. Richtmyer, *Principles of Advanced Mathematical Physics*, (Springer, New York, 1978).
 - [21] A. Ishibashi and A. Hosoya, *Phys. Rev. D* **60**, 104028 (1999).
 - [22] H. Weyl, *Math. Ann.*, **68**, 220, (1910).
 - [23] J. von Neumann, *Math. Ann.*, **102**, 49, (1929).
 - [24] Y. Sucu and N. Ünal, *J. Math. Phys.* **48**, 052503, (2007).
 - [25] J. P. M. Pitelli and P. S. Letelier, *Phys. Rev. D* **77**, 124030, (2008).
 - [26] O. Unver and O. Gurtug, *Phys. Rev. D* **82**, 084016, (2010).
 - [27] S. Habib. Mazharimousavi, O. Gurtug, M. Halilsoy and O. Unver, *Phys. Rev. D* **84**, 124021, (2011).
 - [28] O. Gurtug, M. Halilsoy and S. H. Mazharimousavi, *Advances in High Energy Physics*, 684731, (2015).